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Patrick Leoni

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Patrick Leoni
University of Zurich
Institute for Empirical Research in Economics
Winterthurerstrasse 30, CH-8006 Zurich, Switzerland
E-mail: pleoni@iew.unizh.ch

Abstract

A natural conjecture is that speculative trade disappears when individual beliefs become correct through learning. Sandroni in [22] gives a counterexample in an economy with sunspots. We generalize Sandroni's result by showing that the conjecture holds for economies with complete markets only. We consider a standard finite-horizon General Equilibrium model with complete markets, where uncertainty is represented by fluctuations in individual endowments. Individual beliefs are formed through arbitrary learning processes, and become eventually correct. We show that along every path of events, equilibrium prices of traded assets converge to rational expectations for the sup-norm. We also give a set of sufficient conditions on beliefs and aggregate endowment leading to market crashes, as in Sandroni [22]. We show that such situations are generically continuous perturbations of rational expectations behaviors when beliefs satisfy a requirement introduced here.

Keywords: Asset Pricing, Speculation, Market Crashes

JEL Classification: G12

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1 Introduction

When agents trade financial assets, individual assessment of economic uncertainty is an important factor in determining individual portfolio holdings, and in turn asset prices. Often, heterogeneity of beliefs is identified with speculative trade in the economic literature.[1] A natural and intuitive conjecture is that effects on asset prices of heterogenous individual opinions (and implicitly market psychology), should become negligible as beliefs become correct, for instance through learning or disclosure of financial information.

Surprisingly, Sandroni [22] shows that, when sunspots occur in dynamically complete markets, the conjecture breaks down. In particular, Sandroni gives a simple frictionless example where eventually correct beliefs lead to recurrent market crashes, whereas the above conjecture would predict a no-trade *status-quo* in the long run.

The current paper generalizes Sandroni's result by identifying economic conditions for which the conjecture holds true. We show that such conditions are tight, and that under those conditions market crashes are continuous perturbations of rational expectations situations where market reactions are entirely driven by market fundamentals.[2]

Our contribution identifies which financial structure offsets market psychology to the profit of market fundamentals, allowing in turn to reduce market volatility and crashes as in Sandroni [22].

We show that, in regular economies with complete markets, equilibrium prices of traded securities traded behave continuously as a function of individual beliefs, as beliefs become arbitrarily correct in a sense introduced here. This property ensures convergence towards rational expectations prices, proving in turn the conjecture. Since regular economies are shown to be generic[3] in individual endowments and initial portfolio holdings, we thus establish that the conjecture is true for almost all economies with complete markets.

When economies are not regular, the conjecture still holds, but the continuous dependency between beliefs and equilibrium asset prices cannot be established.

[1] This idea can be found, for instance, in Harris and Raviv [12], Harrison and Kreps [13], Kandel and Pearson [14], Kurz [15], and Morris [20].

[2] See for instance Allen and Jordan [1] for a retrospective of the notion of rational expectations equilibria.

[3] See Debreu [7] for a definition and complete discussion of the concept of genericity.

Consequently, for all economies with complete markets, speculative trade is shown to be less and less influential in terms of asset prices as agents learn, and to eventually disappear when learning processes generate accurate enough predictions.

To explain the origins of market crashes in Sandroni [22], we show that with complete markets and no aggregate uncertainty, market crashes cannot be arbitrarily low. This result implies that arbitrarily low market crashes in Sandroni's are driven by market incompleteness generated by sunspots. In contrast to Cass and Shell [6], this result establishes that sunspots matter because they generate incomplete markets.

Our result does not rule out the possibility of market crashes in economies with complete markets; for instance we show that a significant drop in aggregate endowment largely anticipated by agents leads to a crash with strictly positive probability. Our contribution establishes that such market crashes in (almost all) economies with complete markets are driven by market fundamentals when enough information is available to traders, the market volatility attributed to speculative trade becoming small.

We also advocate the idea that psychological factors can significantly affect markets behaviors. Even though we argue that, when our conditions are met, market behaviors are entirely correlated with market fundamentals, we implicitly identify two situations where psychological factors can matter:

- Lack of accurate information available to the market: in this case, slight variation in public information may generate misleading estimations of actual economic states, and in turn subjectively anticipated economic fluctuations can lead to crashes,

- Incompleteness of markets: the inability to fully hedge against every contingency can lead to large transfer of wealth across agents, as explained in Sandroni [22], leading to volatile markets and possibly crashes.

Overall, this work makes it clear that mandatory disclosure of all private information in financial markets, together with the development of financial tools allowing to hedge against the largest possible set of contingencies, significantly reduce the influence of speculative trades. In turn, large implementation of such recommendations will consequently reduce volatility in financial markets.

In more details, we develop a standard intertemporal general equilibrium model, with finite but large horizon. Uncertainty is represented by random individual endowments. Agents trade future securities among themselves in order to hedge against such randomness. Securities live for one-period ahead only, and are traded in financial markets opened in every period (*sequential markets*). In every period, markets are assumed to be complete (see Leroy and Werner [16] for a definition of this concept).

Every agent has a subjective belief about her future stream of endowments, formed through arbitrary learning processes.[4] Every agent also believes that others' beliefs are uncorrelated with, hence uninformative about, the aggregate endowment process. This last assumption avoids to model hierarchies of beliefs, as described in Mertens and Zamir [18], and the possibility of no trade as in Milgrom and Stockey [19]. Such beliefs can result from endogenous learning, Bayesian learning fits in this framework when the horizon is ever growing but finite.[5] For instance, as in Kurz [15], agents can be endowed before trades with the final posterior generated by their learning experience, leading to beliefs consistent with equilibrium learning and trading. Heterogeneity of beliefs can then be justified by private information or different priors.

Given a subjective belief, every agent is assumed to make consumption-investment decisions so as to maximize the (subjective) expected sum of discounted one-period utility derived from consumption of the good. The one-period utility functions satisfy standard assumptions in financial economics, such as the Inada conditions.

In this framework, we first relate market crashes to individual beliefs and aggregate endowments. Fix $\varepsilon > 0$, an $\varepsilon-$*crash* occurs in a given period when at least one asset has a return below ε in this period (this definition was first partially introduced in Sandroni [22]).

Proposition 2 states that a significant drop in aggregate endowment in a given period, and assigned high probability by all the agents, generates a market crash with strictly positive true probability next period. The intuition is that, when such a scenario is anticipated, demand for hedging assets is high, this in turn leads to high purchasing prices and low returns amplified by the drop in endowment.

[4] We assume that every agent assigns strictly positive probability to every event. This rules out subjective arbitrages as pointed out by Araujo and Sandroni [2].

[5] See Doob [10], Breiman *et al.* [5], and Blackwell and Dubins [4]; see also Diaconis and Freedman [11] for problems arising from an infinite number of states of nature.

Without aggregate uncertainty, Proposition 3 states that crashes cannot be arbitrarily low, no matter how subjective beliefs are formed. Thus, without aggregate uncertainty, speculative trade cannot generate major market crashes without the influence of other economic factors.

Whereas the above results are true for arbitrary learning processes and with possibly infinite horizon, the remainder of the paper analyzes the case where beliefs become arbitrarily correct in finite but large horizon. We say that an agent *learns* if her conditional beliefs along every path of history become arbitrarily close to the true conditional probabilities, the speed at which this phenomena occurs being explicitly controlled.

With this concept, we study convergence of assets prices to rational expectations prices. The following convergence results hold for *regular* economies only; i.e., for economies where aggregate demands for consumption goods and assets after every event are non-degenerate (see Debreu [7] for more details on this issue). This is not problematic from an economic standpoint since, from Proposition 5, every economy is regular (or will become regular) for all but a measure-zero set of aggregate endowments and initial portfolio holdings.

Proposition 8 states that, when the horizon is large enough so that agents have enough time to learn, equilibrium asset prices are arbitrarily close to rational expectations prices for the sup-norm. Moreover, Proposition 8 shows existence of a local diffeomorphism between individual beliefs and equilibrium prices. This has a strong economic intuition, since small variations in individual opinions thus lead to small, and hence controlled, variations in equilibrium prices for accurate enough beliefs.

When the economy is not regular, convergence still obtains for a selection of equilibrium prices only. Continuous dependency between equilibrium prices and beliefs is not always true though.

A related result can be found in Sandroni [21] Proposition 6, with infinite horizon economies. This last result uses a weaker topology of convergence than ours, and continuous variations around rational expectations behaviors are not established. Moreover, asymptotic convergence to rational expectations is not established for every possible history, as we do here.

The paper is organized as follows: in Section 2 we present the model, in Section 3 we formally define and study market crashes, in Section 4 we formally define the concept of accuracy of beliefs and show convergence to rational expectations for regular economies, and finally in the Appendix we give all the technical proofs.

2 The model

In this section, a formal description of the model is given. We first introduce some notations, useful in defining uncertainty.

Consider a finite number T of periods, to which we add a first period 0. The period T is called the *horizon*. In every period t $(1 \leq t \leq T)$, a state is drawn by nature from a set $S = \{1, ..., L\}$, where L is strictly greater than 1. We denote by S^t the $t-$Cartesian product of S.

For every $s_t \in S^t$, a *cylinder* with base on s_t is the set $C(s_t) = \{s \in S^T | s = (s_t, ...)\}$ of all histories whose t initial elements coincide with s_t.

We define the set Γ_t to be the $\sigma-$algebra consisting of all finite unions of cylinders with base on S^t, and Γ_0 to be the trivial σ-algebra. The sequence $(\Gamma_t)_{0 \leq t \leq T}$ generates a filtration, and we define Γ as the $\sigma-$algebra $\underset{t}{\cup}\Gamma_t$.

Consider now any arbitrary probability measure Q on (S^T, Γ), such that $Q(A) > 0$ for every $A \in \Gamma$.

The conditional probability of Q given a finite history $s_t \in S^t$, denoted by Q_{s_t}, is defined for all $A \in \Gamma$ as

$$Q_{s_t}(A) = \frac{Q(A_{s_t})}{Q(C(s_t))},$$

where A_{s_t} is the set of all paths $s \in S^T$ such that $s = (s_t, s')$ and $s' \in S^{T-t}$.

For any vector of probability measure $\widetilde{Q} = (Q^1, ..., Q^I)$ and any history s_t, denote by $\widetilde{Q}_{s_t}$ the vector of conditional beliefs $(Q^1_{s_t}, ..., Q^I_{s_t})$.

The operators E^Q and $E^Q(.|\Gamma_t)(s)$, for every $s = (s_t, ...)$, are the expectation operators associated with Q and Q_{s_t} respectively.

A finite history $s_{t+p} \in S^{t+p}$ *follows* a finite history $s_t \in S^t$, denoted by $s_{t+p} \hookrightarrow s_t$, if there exists $s \in S^p$ such that $s_{t+p} = (s_t, s)$.

We next describe the economy in more details; e.g., preferences, endowments and the assets structure.

2.1 The agents

There are I agents, for some integer $I > 1$, who live for $T + 1$ periods.

There is a single consumption good available in every period t $(0 \leq t \leq T)$. We denote by $c^i_{s_t}$ the consumption of agent i in period t, after the history $s_t \in S^t$.

In every period t, and after every history $s_t \in S^t$, every agent i is endowed with $w^i_{s_t} > 0$ units of consumption goods. The aggregate endowment w_{s_t}, after every event s_t, is thus

$$w_{s_t} = \sum_{i=1,...,I} w^i_{s_t}.$$

In every period t $(0 \leq t \leq T-1)$, and before the realization of the event next period, a new market for securities opens and the agents trade J securities $(J \geq L)$ that live for one period ahead. The supply of each security is normalized to be 0 after every history.

Every security j $(j = 1, ..., J)$, purchased after a history s_{t-1}, payoffs a dividend $d^j_{s_t} \geq 0$ if history $s_t \hookrightarrow s_{t-1}$ is realized, and 0 otherwise. The ex-dividend price of security j purchased after history s_t is denoted by $q^j_{s_t}$. We define the vectors $q_{s_t} = (q^1_{s_t}, ..., q^J_{s_t})$ and $d_{s_t} = (d^1_{s_t}, ..., d^J_{s_t})$.

A *portfolio* $\theta^i_{s_t}$ for every agent i, in history s_t, is a vector of J securities holdings. We set

$$\theta^i = \left(\theta^i_{s_t}\right)_{t \leq T}$$

to be the *portfolio strategy* of agent i. Every agent i has no initial portfolio at date 0, and we use the convention that $\theta_{-1} = 0$.

In every period and after every finite history, nature draws a state of nature according to an arbitrary probability measure P on (S^T, Γ). We assume that $P_{s_t} > 0$ for every s_t.

Every agent i does not know P; however agent i has a subjective belief about nature, represented by a probability measure P^i on (S^T, Γ). We also assume that $P^i_{s_t} > 0$ for every i and every s_t.

Every agent i gets some utility in each period and after any history s_t from consuming the only consumption good present in the economy.

Every agent i ranks all the possible future consumption sequences $c = (c_{s_t})_{s_t \in S^t, 0 \leq t \leq T}$ according to the following utility function:

$$U^i(c) = E^{P^i}\left(\sum_{0 \leq t \leq T} \beta^t u(c_t)\right) \tag{1}$$

where $\beta > 0$ is an intertemporal discount factor, and u is a strictly increasing, strictly concave, twice-continuously differentiable function satisfying the Inada condition, namely $(u)'(c) \mapsto \infty$ as $c \longmapsto 0$ and $(u)'(c) \mapsto 0$ as $c \longmapsto \infty$.

Given an initial portfolio holding θ_{s_t} in history s_t, the budget constraints faced by agent i from this history on are given by

$$\begin{aligned} c_{s_{t+p}} + q_{s_{t+p}}\theta_{s_{t+p}} &\leq w^i{}_{s_{t+p}} + \left(q_{s_{t+p}} + d_{s_{t+p}}\right)\theta_{s_{t+p-1}} && (2) \\ c_{s_{t+p}} &\geq 0 && (3) \end{aligned}$$

for every $s_{t+p} \hookrightarrow s_t$ ($s_{t+p} \in S^{t+p}$ and $0 \leq p \leq T-t$).

For every i ($i = 1, ..., I$), let $B^i_{s_t}(q)$ denote the set of sequences (c, θ) that satisfy conditions (2)-(3) above, for a system of securities prices q and for a particular initial portfolio holding.

We next define the equilibrium concept for this economy.

A *Radner equilibrium* is a sequence of consumption and of portfolio strategy $(c^i, \theta^i)_{1\leq i\leq I}$, and a system of assets prices q such that:

1) For every i, the sequence (c^i, θ^i) maximizes (1) subject to

$$(c^i, \theta^i) \in B^i_0(q), \text{ and}$$

2) markets clear after every history; i.e., for every s_t, we have that

$$w_{s_t} = \sum_{i=1,...,I} c^i_{s_t} \text{ and } \sum_{i=1,...,I} \theta^i_{s_t} = 0.$$

We assume that markets are *complete* for every possible equilibrium vector of asset prices; i.e, for every vector of asset prices q the one-period matrix $\{q_{s_{t+1}} + d_{s_{t+1}}\}_{s_{t+1}\hookrightarrow s_t}$ has rank L, for every finite history s_t.

3 Market crashes

We next describe the notions of *market crashes*, and give a set of sufficient conditions generating them.

For every system of asset prices q, define first the *return* of security j ($j = 1, ..., J$) in history s_{t+1}, when purchased in history s_t, as follows:

$$R^j_{s_{t+1}} = \frac{q^j_{s_{t+1}} + d^j_{s_{t+1}}}{q^j_{s_t}}.$$

Since assets live for one period only, the equilibrium price $q^j_{s_{t+1}}$ in the above must be 0. We use this property throughout.

With this notion, we can describe the meaning of market crash. The following definition has been partially introduced in Sandroni [22].

Definition 1 *For every $\varepsilon > 0$, an ε-crash occurs in history s_t if $R^j_{s_t} < \varepsilon$ for some asset j such that $R^j_{s_t} > 0$.*

We next introduce a narrowed set of securities, allowing us to simplify our analysis. For any $j \in \{1, .., L\}$, define an *Arrow security* $a^j_{s_t}$ to be an asset traded in history s_t that payoffs 1 unit of consumption good if the history (s_t, j) is realized, and 0 otherwise. We denote by $p^j_{s_t}$ the price of the Arrow security $a^j_{s_t}$. Since markets are complete at every equilibrium prices, it is shown in Leroy and Werner [16] Chapter 23 that, for every j ($j = 1, ..., J$),

$$q^j_{s_t} = \sum_{1 \le s \le L} d^j_{(s_t,s)} . p^s_{s_t}. \tag{4}$$

Thus, it is sufficient to find the prices of all Arrow securities to price any asset in this economy. We will use this property in the next two sections to get our results on market crashes.

3.1 Risky aggregate endowments

In this section, we study market crashes under the assumption that aggregate endowment is risky after every history. Beliefs in this section are arbitrary. Even though the model presented so far has a finite horizon, all the results in this section and the next are true with an infinite horizon.

The next result gives a set of sufficient conditions on beliefs and aggregate endowments leading to a market crash.

Proposition 2 *Fix $\varepsilon > 0$ and a history s_t. There exist two strictly positive constants δ and γ such that:*

if there exists a successor s_{t+1} of s_t satisfying both $w_{s_t} > \gamma > w_{s_t+1}$ and $P^i_{s_{t+1}} > \delta$ for every i, then an $\varepsilon-$crash occurs in period $t+1$ with strictly positive true probability.

The above result states that, for a significant drop of aggregate endowment next period assigned high probability by the agents, a market crash can potentially occur. The intuition of this result is that, when expecting future low endowments, agents will increase their demand for securities to hedge against this event. This, in turn, will raise the purchasing price of those securities and therefore lowering their returns.

3.2 Riskless aggregate endowments

In this section, we carry out a similar analysis as in the previous section, under the assumption that aggregate endowments are riskless. In other words, we assume in this section that $w_{s_t} = w$ for every s_t and for some $w > 0$. Beliefs are arbitrary, as in the previous section, and the result below also holds true with an infinite horizon.

The point of this result is to show that arbitrarily low market crashes cannot be driven solely by considerations on beliefs. In contrast, Sandroni [22] shows that, in this case, crashes solely driven by beliefs can occur with complete markets with sunspots. Thus, sunspots should be regarded as generating incomplete markets.

Proposition 3 *There exists $\overline{\varepsilon} > 0$ such that for every $\varepsilon < \overline{\varepsilon}$, an $\varepsilon-$crash occurs with true probability 0 in every period.*

The above result states that assets returns cannot go below a particular level when there is no uncertainty about aggregate endowment. In this case, fluctuation in beliefs cannot drive a market crash.

All the above is true for arbitrary beliefs; the next section analyzes the case where they become correct. It aims to show that, as learning takes place, all the behaviors presented above are continuous perturbations of rational expectations behaviors in regular economies. When horizon are long enough to allow for learning processes to generate accurate predictions (in a sense defined later), market crashes as above are shown to be mostly driven by fundamentals.

4 Convergence to rational expectations

In this section, we show that when beliefs become correct, in a sense defined, equilibrium prices of subsequently traded securities converges towards rational equilibrium prices in regular economies.

We first define the concept of *regular economy.* Define the vector of net aggregate demand for consumption at history s_t, for asset prices q and individual beliefs $\tilde{P} = (\tilde{P}^1, ..., \widetilde{P}^I)$, to be

$$C_{s_t}(q, \widetilde{P}) = \sum_{i=1,...,I} c^i_{s_t}(q, \widetilde{P}^i) - w_{s_t}.$$

Define also, for every history s_t, the vector of aggregate demand for assets at prices q and beliefs $\widetilde{P}$ to be

$$\theta_{s_t}(q, \widetilde{P}) = \sum_{i=1,\ldots,I} \theta^i_{s_t}(q, \widetilde{P}^i).$$

Fix a horizon T, and consider the vector $\Theta(q, \widetilde{P})$ formed with all the above demands functions for every possible history. By definition of Radner equilibria, a system of prices q is an equilibrium system of prices, at beliefs $\widetilde{P}$, if $\Theta(q, \widetilde{P}) = 0$.

Consider an economy with horizon T, where every agent agrees with the true; i.e., every agent has the correct belief P. Every Radner equilibrium in such an economy is called a *rational expectations equilibrium.*

We next define the concept of *regular economies.*

Definition 4 *Consider correct beliefs $\overline{P} = (P, ..., P)$ and a system of prices q such that $\Theta(q, \overline{P}) = 0$. An economy is regular at $(q, \overline{P})$ if $D_q\Theta(q, \overline{P})$ has full rank.*

The above definition requires the economy to be well-behaved at rational expectations beliefs only.

Another relevant concept for this paper is the concept of *local economy.* We call a local economy starting at history s_t, an economy with initial history s_t, whose future histories are all successors of s_t and endowments in those histories are identical to the original economy, and with an initial portfolio holdings in period s_t.

Within this local economy, agents with individual portfolio holdings in s_t trade assets so as to maximize expected sum of discounted utility derived from consumption in subsequent histories, where expectations are taken conditional on reaching s_t.

We then say that an economy is *locally regular* after s_t if the local economy starting at s_t is regular.

Regular economies are generic in initial portfolio holdings and endowments. This result is stated in the next proposition.

Proposition 5 *For almost every level of endowments and for almost every initial portfolio holdings, an economy is regular.*

The above result states that, with probability one with respect to the fundamentals, an economy is regular. A similar result, in the context of one-period economies only, can be found in Balasko [3] and in Debreu [8]. Also, the above result extends easily to locally regular economies; i.e., for almost every level of endowments and for almost every initial portfolio holdings after a particular history, an economy is locally regular at this history.

Next is defined a concept of convergence of beliefs. From now, the horizon is assumed to be expanding; i.e., the horizon T is no longer assumed to be fixed, but will remain finite.

Our notion of accuracy of predictions is inspired from Definition 2 in Sandroni [21]. It captures some measure of closeness between conditional individual belief, regarded as learning process, and true conditional beliefs as more information is available to an agent over time.

We define the sup-norm over the space of probability measures. For two probability measures P and Q defined on (S^T, Γ), we set

$$\|P - Q\| = \max_{A \in \Gamma} |P(A) - Q(A)|.$$

Definition 6 *Consider a sequence of real numbers $\alpha = (\alpha_t)_t$ converging to 0, and individual belief P^i for some agent i. Fix also a path s. Agent i learns α-fast along the path s if for every $p \in \mathbb{N}$ and every horizon $T \geq p$*

$$\|P^i_{s_{T-p}} - P_{s_{T-p}}\| \leq \alpha_{T-p}.$$

Agent i learns α-fast if she learns α-fast along every path s.

In the above definition, an agent learns if her conditional beliefs become arbitrarily close to the true conditional distribution over the states of nature, the speed at which this phenomena occurs being explicitly controlled.

Even though the next results are stated for regular economies, they extend to locally regular economies in a natural manner.

The next proposition is central to the paper. It establishes that, when beliefs are accurate enough, Radner equilibrium prices and individual beliefs become diffeomorphic in regular economies.

Denote by W_{s_t} the vector of individual endowments in the local economy starting at s_t, with the convention that individual portfolio holdings in s_t are regarded as endowments.

Proposition 7 *Assume that the economy is regular. For every local economy starting after any history* s_t*, there exist an open neighborhood* $\mathcal{P}_{s_t}$ *of* $\overline{P}_{s_t}$ *and* W_{s_t} *(in the cartesian product of space of conditional beliefs and individual endowments), an open neighborhood* $\mathcal{Q}_{s_t}$ *of rational expectations equilibrium prices, and a unique diffeormorphism* g_{s_t} *such that, for every* $(\widetilde{P}, W) \in \mathcal{P}_{s_t}$*, the vector* $g_{s_t}(\widetilde{P}, W)$ *is a vector of equilibrium prices.*

Before stating the main result of the paper, we first define a norm on the space of conditional beliefs. Consider two vectors of individual beliefs $\widetilde{P} = (P^1, ..., P^I)$ and $\widetilde{Q} = (Q^1, ..., Q^I)$, where every component of the previous sequence defines a probability measure on (S^T, Γ). Fix a history s_t, and define the *conditional sup-norm* to be

$$\left\| \widetilde{P} - \widetilde{Q} \right\|_{s_t} = \max_i \max_{A \in \Gamma} \left| P^i_{s_t}(A) - Q^i_{s_t}(A) \right|$$

For the next result only, we make the additional assumption that there exists a constant $B > 0$ such that

$$w_{s_t} < B \text{ for every } s_t.$$

In words, aggregate endowments are now assumed to be uniformly bounded.

Denote first by $\overline{q}^j_{s_t}$ (resp. $q^j_{s_t}$) the equilibrium price of security j purchased after history s_t, associated with correct beliefs $\overline{P}$ (resp. with subjective beliefs $\widetilde{P}$).

Proposition 8 *Assume that the economy is regular for every finite horizon. There exists a sequence of reals* α *converging to 0 such that, if every agent learns* α*-fast then for every path* s*, for every* $p \in \mathbb{N}$ *and for every asset* j

$$|q^j_{s_{T-p}} - \overline{q}^j_{s_{T-p}}| \to 0 \textit{ as } T \textit{ converges to } +\infty.$$

The above result states that, when there is enough time for learning processes to generate accurate predictions, Radner prices are arbitrarily good approximations of rational expectations prices towards the end of the horizon.

Regularity is critical to ensure convergence as described in the above: the issue is that outside of regular economies there may be no locally isolated equilibrium prices and selection of some particular equilibrium prices is needed.

5 Appendix

In this appendix, we prove all the results presented earlier in the analysis.

5.1 Proof of Proposition 2

The strategy of our proof goes as follows. We first redefine equilibrium asset returns in terms of equilibrium prices of Arrow securities, and then we show that at least one such Arrow security has an arbitrarily high price when some conditions on beliefs and aggregate endowments are met. This will lead to an arbitrarily low return on at least one asset, and in turn to a market crash.

Fix $\varepsilon > 0$ and fix a history s_t.

By (4) and our remark on Arrow security prices at the beginning of Section 3, the equilibrium return in $s_{t+1} \hookrightarrow s_t$ of a security j, purchased in history s_t, is given by

$$R^j_{s_{t+1}} = \frac{d^j_{s_{t+1}}}{\sum\limits_{1 \leq s \leq L} d^j_{(s_t,s)} . p^s_{s_t}}, \tag{5}$$

where $p^s_{s_t}$ is the equilibrium price of the Arrow security $a^s_{s_t}$ for every s.

We now find Arrow security prices. Consider an asset structured entirely composed of Arrow securities instead of the original asset structure. Since markets are complete, such asset structure can be derived from a replication of existing securities, validating our approach.

For this asset structure, consider the program faced by any agent i, namely maximizing (1) subject to

$$\begin{aligned} c_{s_t} + \overline{p}_{s_t} \theta_{s_t} &\leq w^i_{s_t} + \overline{d}_{s_t} \theta_{s_{t-1}} \\ c_{s_t} &\geq 0 \end{aligned}$$

for every s_t, where $\overline{p}_{s_t} = (p^s_{s_t})_{s=1,\ldots,L}$ is the vector of Arrow security prices, θ_{s_t} is the vector of holding of such securities, and $\overline{d}_{s_t}$ their payoff.

The first-order conditions to this program give, for every s and i,

$$\beta . P^i_{(s_t,s)} . u'(c^i_{(s_t,s)}) = p^s_{s_t} . u'(c^i_{s_t}). \tag{6}$$

It is easy to see that, for every $\delta > 0$ such that $P^i_{(s_t,s)} > \delta$ for every i, there exists an agent, denoted by $\delta(i)$, such that $c^{\delta(i)}_{s_t} \geq \frac{w_{s_t}}{I}$ in equilibrium.

This last remark implies that, since u satisfies the Inada conditions, the expression $u'(c_{s_t}^{\delta(i)})$ is bounded away from $+\infty$ as δ converges to 1.

Also, since $c_{(s_t,s)}^{\delta(i)} \leq w_{(s_t,s)}$ and by the Inada conditions, a low enough value of aggregate endowment $w_{(s_t,s)}$ in history (s_t,s) will increase the left-hand side of (6) above to an arbitrary high level. Thus, as δ converges to 1 and $w_{(s_t,s)}$ converges to 0, for (6) to hold for agent $\delta(i)$ it must be true that $p_{s_t}^s$ converges to ∞. Thus, one can find the two constants δ and γ leading to an arbitrary high price $p_{s_t}^s$.

Since markets are complete, there exists an asset giving a strictly positive dividend in history (s_t, s). By using (5) and our previous remark, we have thus shown that the return of this asset can be made arbitrarily low for the appropriate values of δ and γ, leading to an $\varepsilon-$crash in history (s_t, s).

Since history (s_t, s) occurs with true strictly positive probability, the proof is now complete.

5.2 Proof of Proposition 3

The strategy of the proof is as follows. We carry out our analysis with Arrow securities, to show that their equilibrium prices are uniformly bounded from infinity. Equation (5) will then allow us to conclude.

Consider an asset structure entirely composed of Arrow securities, and any history s_t. Consider also any history $s_{t+1} = (s_t, s)$ following s_t.

Notice first that, there always exists an agent i_0 whose consumption in history s_{t+1} is such that $c_{s_{t+1}}^{i_0} \geq \frac{w}{I}$. Since u' is strictly decreasing, we thus have that $u'(c_{s_{t+1}}^{i_0}) \leq u'(\frac{w}{I})$.

Consider then (6), for agent i_0. It follows from the above remark that

$$\begin{aligned} p_{s_t}^s.u'(c_{s_t}^{i_0}) &= \beta.P_{s_{t+1}}^{i_0}.u'(c_{s_{t+1}}^{i_0}) \\ &\leq \beta.P_{s_{t+1}}^{i_0}.u'(\frac{w}{I}) \\ &\leq \beta u'(\frac{w}{I}). \end{aligned}$$

Since we also have that $c_{s_t}^{i_0} \leq w$, we derive from the above that for every $s_{t+1} = (s_t, s)$

$$p_{s_t}^s \leq \beta \frac{u'(\frac{w}{I})}{u'(w)}.$$

Thus, we have found an uniform upper-bound on the equilibrium prices of every Arrow security.

By (5), it follows that the equilibrium return of every traded security after every history is uniformly bounded below.

This completes the proof.

5.3 Proof of Proposition 5

The first lemma, central to proving Proposition 5, states that the original framework is equivalent to a new framework where every economic decisions are made in period 0.

The proof of the next lemma is similar to the proof of equivalence between Arrow-Debreu equilibria and Radner equilibria.

Consider first the following environment. Every agent i ($i = 1, ..., I$) has the utility function as in (1), and faces the budget constraints

$$\sum_{s_t,t} p_{s_t} c_{s_t} \leq \sum_{s_t,t} p_{s_t} w^i_{s_t}, \text{ and} \tag{7}$$

$$c_{s_t} \geq 0 \text{ for every } s_t. \tag{8}$$

for a given sequence of consumption prices p. An *Arrow-Debreu equilibrium* is then a sequence of consumption prices p, a sequence of consumption c^i for every i, such that:

1) for every i, the sequence c^i maximizes (1) subject to (7) and (8), at prices p, and

2) markets clear after every history; i.e., for every s_t we have that $w_{s_t} = \sum_{i=1,...,I} c^i_{s_t}$.

Without loss of generality, We normalize prices so that $p_0 = 1$. The next lemma shows that the above setting is equivalent to the original one. This is actually Lemma 1 in Sandroni [21].

Lemma 9 *Consider an Arrow-Debreu equilibrium $(p, (c^i)_{i=1,...,I})$. Then there exists a sequence of portfolio $(\theta^i)_{i=1,...,I}$ and a system of prices q such that $(q, c^i, \theta^i)_{i=1,...,I}$ is a Radner equilibrium.*

Conversely, consider a Radner equilibrium $(q, c^i, \theta^i)_{i=1,...,I}$. Then there exists a sequence of Arrow-Debreu prices p such that $(p, (c^i)_{i=1,...,I})$ is an Arrow-Debreu equilibrium.

It is straightforward to derive a local version of the above lemma; i.e., for local economies starting at any particular history. The only difference is that an adjustment is needed in Arrow-Debreu individual wealth to take account of the net present value of initial portfolio holdings (see Sandroni [21] for more details, this issue is omitted here for sake of simplicity).

Every argument is now made in the Arrow-Debreu framework above, Lemma 9 showing how to switch back to the original setting.

The strategy of the proof is identical to that of standard genericity with commodities markets in an Arrow-Debreu framework (see for for instance Mas-Colell *et al.* [17] Section 17.D). Commodities consumption plans are simply replaced here by contingent consumption plans.

We next outline this proof. The first step is to apply the well-known Transversality Theorem to the vector of net aggregate demand functions.

This theorem is stated next. Consider a system of M equations and N unknown, depending on some parameters $q = (q_1, ..., q_S) \in \mathbb{R}^S$ and solving $f(v_1, ..., v_N; q) = 0$. The function f is assumed to be continuously differentiable.

Theorem 10 (Transversality Theorem) *If the $M \times (N+S)$ matrix $Df(v;q)$ has rank M whenever $f(v;q) = 0$, then for almost every q the $M \times N$ matrix $D_v f(v;q)$ has rank M whenever $f(v;q) = 0$.*

Proof. See for instance Mas-Colell *et al.* [17] Proposition 17.D.3. ■

Denote by $z(p, w)$ the vector of net aggregate demands at correct beliefs $\overline{P}$ for Arrow-Debreu prices p and aggregate endowments w. For sake of simplicity, we omit the implicit dependency of $z(p, w)$ on individual correct beliefs.

Following the same lines as Mas-Colell *et al.* [17] Proposition 17.D.4 or Debreu [7], we can establish that, for every system of Arrow-Debreu prices p and every aggregate endowment w, the rank of $D_w z(p, w)$ is L^T.

Proposition 5 is then established by combining the above fact and the Transversality Theorem.

5.4 Proof of Proposition 7

All the next results are proved with the original framework with sequential markets. To prove Proposition 7, we show that when beliefs are accurate enough, the Implicit Function Theorem applies to our framework and provides existence of the diffeomorphism.

Fix any history s_t, and consider the local economy starting after s_t. Consider any Radner equilibrium $(c_{s_t}, \theta_{s_t}, q_{s_t})_{t\geq 0}$ for the original economy with horizon T.

By the Law of Iterated Expectations, the sequence $(c_{s_{t'}}, \theta_{s_{t'}}, q_{s_{t'}})_{t'\geq t}$ is also a Radner equilibrium for the local economy starting at history s_t with horizon p, where individual beliefs $\widetilde{P}_{s_t}$ are initial beliefs conditional on reaching s_t.

Consider also the restriction of the demand matrix $\Theta(.,.)$ to this local economy, denoted by $\Theta_{s_t}(.,.)$.

Form the above remark, we thus have that

$$\Theta_{s_t}(q^1_{s_t}, ..., q^1_{s_T}, ..., q^J_{s_t}, ..., q^J_{s_T}, \widetilde{P}_{s_t}, W_{s_t}) = 0.$$

By our assumptions on u, and following standard arguments in general equilibrium theory (see for instance Mas-Colell *et al.* [17] Section 17.G or Debreu [8]), the function Θ_{s_t} is C^1. By hypothesis, it is also regular at correct beliefs $\overline{P}_{s_t}$ and corresponding equilibrium prices $(\overline{q}_{s_{t'}})_{t'\geq t}$.

By the Implicit Function Theorem (see Dieudonné [9] Chapter X), there exist an open neighborhood $\mathcal{P}_{s_t}$ of $\overline{P}_{s_t}$ and W_{s_t} (in the cartesian product of space of conditional beliefs and individual endowments), an open neighborhood $\mathcal{Q}_{s_t}$ of rational expectations equilibrium prices, and a unique function g_{s_t} such that

- $g_{s_t}(\overline{P}_{s_t}, W_{s_t}) = (\overline{q}_{s_{t'}})_{t\geq t'}$,
- g_{s_t} is a diffeomorphism between $\mathcal{P}_{s_t}$ and $\mathcal{Q}_{s_t}$, and
- for every $(\widetilde{P}, W) \in \mathcal{P}_{s_t}$ we have that

$$\Theta_{s_t}\left(g_{s_t}(\widetilde{P}, W), \widetilde{P}, W\right) = 0.$$

The diffeomorphism g_{s_t} thus has all the desired properties, and the proof of Proposition 7 is now complete.

5.5 Proof of Proposition 8

The proof proceeds by way of contradiction, and the regularity the diffeomorphisms found in Proposition 7 allows us to derive a contradiction.

Assume that every for sequence α of real numbers converging to 0, every agent learns α-fast and there exist a path s, an integer p, a security j and a constant $\alpha > 0$ such that

$$|q^j_{s_{T-p}} - \overline{q}^j_{s_{T-p}}| > \alpha \text{ for every T,}$$

modulo an omitted extraction of a subsequence from $(s_{T-p})_{T \geq 0}$.

For the path s and integer p, consider any period $t = T - p$ where $T \geq p$. Let $g_{s_t} : \mathcal{P}_{s_t} \rightarrow \mathcal{Q}_{s_t}$ be the unique diffeomorphism associated with s_t by Proposition 7.

We next restrict the sets $\mathcal{P}_{s_t}$ to generate a global diffeomorphism. We show that this diffeomorphism has uniformly bounded variations at correct beliefs and at all actual endowments, and we use this property to derive a contradiction.

To restrict the sets $\mathcal{P}_{s_t}$, we proceed in three steps:

1. since individual endowments and initial portfolio holdings are bounded away from infinity, we restrict the sets $\mathcal{P}_{s_t}$ to be uniformly bounded and to contain $(\overline{P}, W_{s_t})$,

2. for every t, consider the set $U_t = \{t' \mid \mathcal{P}_{s_t} \cap \mathcal{P}_{s_{t'}} \neq \emptyset\}$. If $U_t = \emptyset$, define $\mathcal{U}_{s_t} = \mathcal{P}_{s_t}$. Otherwise, define $\mathcal{U}_{s_t} = \bigcup_{t' \in U_t} \mathcal{P}_{s_{t'}}$,

3. for every t, consider the set $V_t = \{t' \mid cl(\mathcal{U}_{s_t}) \cap cl(\mathcal{U}_{s_{t'}}) \neq \emptyset\}$, where $cl(.)$ denotes the topological closure. If $V_t = \emptyset$, define $\mathcal{V}_{s_t} = \mathcal{U}_{s_t}$. Otherwise, there exists an open ball $\mathcal{B}_t$ containing all $(\overline{P}, W_{s_p}) \in \mathcal{U}_{s_t}$ (where W_{s_p} are actual endowments) such that $\mathcal{B}_t \cap \mathcal{U}_{s_{t'}} \neq \emptyset$ for every $t' \in V_t$. Define then $\mathcal{V}_{s_t} = \mathcal{B}_t$.

Defining $\mathcal{V} = \bigcup_t \mathcal{V}_{s_t}$, we now introduce the function g mapping $\mathcal{V}$ into the set of asset prices, such that

$$g(P, W) = g_{s_t}(P, W) \text{ if } (P, W) \in \mathcal{P}_{s_t}.$$

If some (P, W) belongs to two different sets $\mathcal{P}_{s_t}$ and $\mathcal{P}_{s_{t'}}$, local uniqueness of the diffeomorphisms makes the choice of g_{s_t} and $g_{s_{t'}}$ equivalent.

It is straightforward to show that g is a diffeomorphism between $\mathcal{V}$ and $g(\mathcal{V})$, since the functions g_{s_t} are uniquely defined on $\mathcal{V}_{s_t}$ for every t.

Moreover, the set $\mathcal{V}$ is bounded above and below by construction. Together with the regularity of g, this implies that there exists $C > 0$ such that

$$\sup_{(\overline{P},W)\in\mathcal{V}} \|Dg(\overline{P}, W)\|_* < C, \tag{9}$$

where $\|.\|_*$ is any norm on the image of the partial derivatives of g.

We next use this property to derive a contradiction. Since $\mathcal{V}_{s_t}$ is an open set, there exists $\alpha_t > 0$ such that the open ball of center $(\overline{P}, W_{s_t})$ and radius α_t is included in $\mathcal{V}_{s_t}$.

Consider now the sequence $= (\alpha_t)_t$, which can be assumed to converge to 0 without loss of generality, and any agent i learning α-fast. Let g^j denote the projection of g on the subspace of prices for asset j.

We thus have for every $t \geq p$ that

$$|q^j_{s_t} - \overline{q}^j_{s_t}| = |g^j(\widetilde{P}_{s_t}, W_{s_t}) - g^j(\overline{P}_{s_t}, W_{s_t})| > \alpha.$$

Moreover, since agent i learns α-fast along the path s, we also have that

$$\frac{|g^j(\widetilde{P}_{s_t}, W_{s_t}) - g^j(\overline{P}_{s_t}, W_{s_t})|}{|\widetilde{P}^i_{s_t} - \overline{P}_{s_t}|} > \frac{\alpha}{|\widetilde{P}^i_{s_t} - \overline{P}_{s_t}|} \to \infty$$

as t converges to ∞.

By the continuity of Dg, this last remark implies that the partial derivatives of g^j with respect to individual belief $\widetilde{P}^i$, evaluated at $(\overline{P}, W_{s_t})$, converge to infinity as t converges to infinity.

This is a contradiction to (9), and the proof is now complete.

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